Noncommutative complex analytic spaces

Frank Schuhmacher

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Abstract

In this paper, we define NC complex spaces as complex spaces together with a structure sheaf of associative algebras in such a way that the abelization of the structure sheaf is the sheaf of holomorphic functions.

Introduction

We want to study noncommutative structures on underlying classical geometric objects. A few words to motivate this objective: Let (spaces) be one of the following categories: (schemes), (varieties), (analytic spaces), (smooth manifolds) or an appropriate subcategory of one of them. In particular, (spaces) is a subcategory of the category of locally ringed spaces, i.e. a space consists of a topological space X and a structure sheaf $\mathcal{O}$ of commutative rings and perhaps of extra data. Deformation quantization leads to the problem of finding an extension (NC spaces) of the category (spaces), such that for a space $(X, \mathcal{O})$ and a non commutatively deformed structure sheaf $\mathcal{O}'$ with abelization $\mathcal{O}$, the pair $(X, \mathcal{O}')$ still belongs to (NC spaces). More generally, if $\mathcal{O}'$ is a fibre of a deformation of the structure sheaf $\mathcal{O}$ of X, there should should be an object $(X', \mathcal{O}')$ of (NC spaces), where X' is obtained by a classical deformation of X.

Kapranov [2] has defined an extension (NC schemes) of the category (schemes) in which at least formal noncommutative deformations of the structure sheaf of a scheme live. In this paper, we construct such an extension (NC complex analytic spaces) of the category of complex analytic spaces. An advantage of the analytic context is that we can allow *local* noncommutative deformations, i.e. deformations parametrized by a space germ, not only *formal* deformations, within the category (NC complex analytic spaces). This paper only contains the definition and some examples of NC complex analytic spaces. Their deformations will be described in a forthcomming paper.

In the first section, we give the prerequisites on noncommutative power series and introduce “convergent” noncommutative power series. In the second section, we construct a natural noncommutative structure sheaf on the n-dimensional affine complex space and show that local models of analytic spaces can be equipped with a noncommutative structure sheaf. NC complex analytic spaces are just defined as the

objects obtained by glueing such NC local models. In the third section, we give several examples of NC complex analytic spaces, like supermanifolds and NC projective varieties.

1 Noncommutative power series

The starting point of the theory of NC complex spaces is to replace power series where-ever they occur in the theory of holomorphic functions by noncommutative power series.

Multi-indices For each multi-index $I = (i_1, \dots, i_k)$ in $\{1, \dots, n\}^k$, set

$$\begin{aligned} x^I &:= x_{i_1} \cdot \ldots \cdot x_{i_k}, \\ \#I &:= k. \end{aligned}$$

By definition, there is an empty multi-index 0 with $x^0 = 1$, $\#0 = 0$. For multi-indice I, J, set $I + J := (i_1, \dots, i_{\#I}, j_1, \dots, j_{\#J})$. Let $I < J$ be the relation $\#I < \#J$ and I is obtained from J by deletion of indices. Observe that, if $I \leq J$ (i.e. $I < J$ or $I = J$), then there exists an order-preserving, injective map α from the set $\{1, \dots, \#I\}$ into $\{1, \dots, \#J\}$ such that $i_k = j_{\alpha(k)}$, for $k = 1, \dots, \#I$. We say that α is a map from I to J. Write $\binom{J}{I}$ for the number of such maps $\alpha : I \longrightarrow J$. By definition, $\binom{J}{0} = 1$, for any J. Obviously, $\binom{J}{I} \leq \binom{\#J}{\#I}$. Set $J -_\alpha I$ to be the multi-index obtained from J by deletion of $j_{\alpha(1)}, \dots, j_{\alpha(\#I)}$. For elements $p \in \mathbb{C}^n$, we can just write p^{J-I} insetad of $p^{J-_\alpha I}$, since this value does not depend on the map α. A multi-index I is called **ordered**, if $i_k \leq i_l$, for $k < l$.

Formal and convergent power series For an element $p = (p_1, \dots, p_n) \in \mathbb{C}^n$, we write $\mathbb{C}[[x_1 - p_1 | \dots | x_n - p_n]]$ for the noncommutative power series algebra in formal variables $x_i - p_i$ and $\mathbb{C}[x_1 - p_1 | \dots | x_n - p_n]$ for the subalgebra of noncommutative polynomials. We have a canonical epimorphism ab from $\mathbb{C}[[x_1 - p_1 | \dots | x_n - p_n]]$ to the commutative power series algebra $\mathbb{C}[[x_1 - p_1, \dots, x_n - p_n]]$. Set $f_{\mathrm{ab}} := \mathrm{ab}(f)$. A noncommutative power series

$$f = \sum_I a_I (x - p)^I = \sum_{m=0}^{\infty} \sum_{\#I = m} a_I (x - p)^I$$

in $\mathbb{C}[[x_1 - p_1 | \dots | x_n - p_n]]$ is called **convergent**, if the image f_{abs} of $\sum_I |a_I| (x - p)^I$ under ab is convergent. Write $\mathbb{C}\{x_1 - p_1 | \dots | x_n - p_n\}$ for the algebra of convergent noncommutative power series, and for $r \in \mathbb{R}^n_{>0}$, write $\mathbb{C}_r\{x_1 - p_1 | \dots | x_n - p_n\}$ for the subalgebra consisting of those f such that f_{abs} converges on the open polydisk $P(p, r)$ with multiradius r centered at p. Given such an f, for elements q in $P(p, r)$, set $f(q) := f_{ab}(q)$. Lets make the convention that $\mathbb{C}\{x - p\}$ stands for $\mathbb{C}\{x_1 - p_1 | \dots | x_n - p_n\}$

(always with n variables) and that $\mathbb{C}\{y-q\}$ stands for $\mathbb{C}\{y_1-q_1|\dots|y_m-q_m\}$ (always with m variables).
A noncommutative power series $f=\sum_I a_I(x-p)^I$ is called **commlike**, if $a_I=0$, for nonordered I. The map ab from the noncommutative to the commutative power series ring has a splitting unab in the category of $\mathbb{C}$-vectorspaces, such that the composition unab $\circ$ ab fixes exactly the commlike power series.

Morphisms If $g=\sum_J b_J(x-p)^J$ is a power series in $\mathbb{C}[[x-p]]$ and if the power series $f_1,\dots,f_m$ with $f_\nu=\sum_I a_{\nu,I}(y-q)^I$ belong to the maximal ideal $\mathfrak{m}_{[[y-q]]}$ of $\mathbb{C}[[y-q]]$, we can form the power series

$$g(f_1,...,f_m):=\sum_J(\sum b_K\cdot a_{k_1,I_1}\cdot\ldots\cdot a_{k_{k_{\#K}},I_{k,\#K}})(y-q)^J,$$

where the sum in the bracket is over all multi-indices K such that $J=I_{k_1}+\ldots+I_{k_{\#K}}$.
A **morphism** $\mathbb{C}[[x-p]]\longrightarrow\mathbb{C}[[y-q]]$ of noncommutative power series algebras is a local algebra homomorphism of the form $g\mapsto g(f_1,\dots,f_n)$, for a given n-tuple $(f_1,\dots,f_n)$ of elements of $\mathfrak{m}_{[[y-q]]}$.

Proposition 1.1. *If g and each f_i are convergent power series, then $g(f_1,\dots,f_m)$ is also convergent.*

A **morphism** $\mathbb{C}\{x-p\}\longrightarrow\mathbb{C}\{y-q\}$ of convergent noncommutative power series algebras is a local algebra homomorphism of the form $g\mapsto g(f_1,\dots,f_n)$, for a given n-tuple $(f_1,\dots,f_n)$ of elements of $\mathfrak{m}_{\{y-q\}}$.

Proposition 1.2. *Let $f:\mathbb{C}[[x-p]]\longrightarrow\mathbb{C}[[y-q]]$ and $g:\mathbb{C}[[y-q]]\longrightarrow\mathbb{C}[[z-r]]$ be given by $f(x_\nu-p_\nu)=\sum_I a_{\nu,I}(y-q)^I$ and $g(y_\mu-q_\mu)=\sum_J b_{\mu,J}(z-r)^J$. Then, we have $g(f(x_\nu-p_\nu))=\sum_K c_K(z-r)^K$, with*

$$c_K=\sum_I a_{\nu,I}\sum b_{i_1,J_1}\cdot\ldots\cdot b_{i_{\#I},J_{\#I}},$$

where the second sum is taken over all multi-indices $J_1,\dots,J_{\#I}$ such that $J_1+\ldots+J_{\#I}=K$.

For an endomorphism f of $\mathbb{C}[[x-p]]$, with $f(x_\nu-p_\nu)=\sum_I a_{\nu,I}(x-p)^I$, set Jf to be the $n\times n$-matrix $(a_{\nu,i})_{\nu,i}$.

Theorem 1.3. *An endomorphism f of $\mathbb{C}[[x-p]]$ is an automorphism, if and only if Jf is invertible.*

Proof. Suppose that Jf is invertible. Inductively, for $k\ge 1$, $\mu=1,\dots,n$ and $J\in\{1,\dots,n\}^\mu$, we will define coefficients $b_{\mu,J}\in\mathbb{C}$ such that the endomorphism $g^{(k)}$ of $\mathbb{C}[[x-p]]$ with $g^{(k)}(x_\mu-p_\mu)=\sum_{\#J\le k}b_{\mu,J}(x-p)^J$ is inverse to f modulo $\mathfrak{m}^{k+1}_{[[x-p]]}$.

For $k = 1$, by Proposition 1.1, the necessary (and sufficient) conditions are

$$\sum_{i=1}^{n} a_{\nu,i} b_{i,\nu} = 1 \quad \text{for all } \nu,$$
$$\sum_{1=1}^{n} a_{\nu,i} b_{i,\mu} = 0 \quad \text{for all } \nu \neq \mu.$$

We can find such coefficients, if and only if Jf is invertible. Now suppose that Jf is invertible and that the coefficients $b_{\mu,J}$ are constructed adequately, for $\#J \leq k-1$. For fixed J with $\#J = k$, we have to find $b_{\mu,J}$ such that

$$\sum_{i=1}^{n} a_{\nu,i} b_{i,J} + \sum_{\#I \geq 2} a_{\mu,I} \sum b_{i_1,J_1} \cdot \ldots \cdot b_{i_{\#I},J_{\#I}} = 0.$$

The second sum is known, and since Jf is invertible, we can find adequate $b_{1,J}, \ldots, b_{n,J}$. $\square$

Corollary 1.4. *Each lift of an automorphism of a commuative (convergent) power series ring to an endomorphism of the noncommutative (convergent) power series ring is again an automorphism.*

Complete tensor products For power series algebras $\mathbb{C}[[x]] = \mathbb{C}[[x_1|...|x_n]]$ and $\mathbb{C}[[y]] = \mathbb{C}[[y_1|...|y_m]]$, we define the **free product**

$$\mathbb{C}[[x]] * \mathbb{C}[[y]] := \mathbb{C}[[x_1|...|x_n|y_1|...|y_n]]$$

and the **analytic tensor product** $\mathbb{C}[[x]]\hat{\otimes}_{\mathbb{C}}\mathbb{C}[[y]]$ as the power series algebra in $x_1, ..., x_n$ and $y_1, ..., y_m$, where the y_j are assumed to commute with the x_i. We make the corresponding definitions for convergent power series algebras.

Finitely generated ideals In the NC analytic context, the concept of finitely generated two-sided ideals has to be slightly adapted. As a reason, we give the following example:

Example 1.5. Let (x) be the two-sided ideal of $\mathbb{C}[[x|y]]$, consisting of all noncommutative power series where at least one factor x arises in each monomial. Observe that (x) is not the two-sided ideal generated by x in the algebraic sense.

Proof. Assume that (x) is the two-sided ideal generated by x in the algebraic sense. Then we can find power series $f_i = \sum_j a_{ij} y^j$ and $g_i = \sum_j b_{ij} y^j$ in $\mathbb{C}[[y]]$, $i = 1, \ldots, N$ such that

$$xyx + y^2xy^2 + ... = \sum_{i=1}^{N} f_i(y) \cdot x \cdot g_i(y).$$

The right hand-side takes the form $\sum_{j,k}(\sum_i a_{ij}b_{ik})y^j xy^k$. In particular, for $j,k \leq N+1$, we would get $\sum_i^N a_{ij}b_{ik} = \delta_{j,k}$, which is impossible, since the left hand-side is a product of two matrices of rank at most N. $\square$

By definition, the opposite algebra $\mathbb{C}[[x]]^{\text{op}}$ $\{f^{\text{op}}: f \in \mathbb{C}[[x]]\}$ of $\mathbb{C}[[x]]$ is the set $\{f^{\text{op}}: f \in \mathbb{C}[[x]]\}$ with operations $f_1^{\text{op}} + f_2^{\text{op}} = (f_1 + f_2)^{\text{op}}$ and $f_1^{\text{op}} \cdot f_2^{\text{op}} = (f_2 \cdot f_1)^{\text{op}}$. Observe that $\mathbb{C}[[x]]^{\text{op}}$ is naturally isomorphic to the power series algebra $\mathbb{C}[[x_1^{\text{op}}|...|x_n^{\text{op}}]]$ and the assignment $x \mapsto x^{\text{op}}$ defines an isomorphism $\text{OP}: \mathbb{C}[[x]] \longrightarrow \mathbb{C}[[x]]^{\text{op}}$. Attention, in general, $\text{OP}(f) \neq f^{\text{op}}$, for example, $\text{OP}(x_1x_2) = x_1^{\text{op}}x_2^{\text{op}} = (x_2x_1)^{\text{op}}$. We define the **(complete) envelopping algebra** $\mathbb{C}[[x]]^{\hat{e}}$ of $\mathbb{C}[[x]]$ as $\mathbb{C}[[x]]\hat{\otimes}_{\mathbb{C}}\mathbb{C}[[x]]^{\text{op}}$. Consider the natural epimorphism

$$\hat{\alpha}: \mathbb{C}[[x]]^{\hat{e}} \longrightarrow \mathbb{C}[[x]].$$

For a two-sided ideal $J \subseteq \mathbb{C}[[x]]$, the inverse image $\hat{\alpha}^{-1}(J)$ is not, in general a left ideal of $\mathbb{C}[[x]]^{\hat{e}}$, since it is not, in general, closed under left multiplication by elements of $\mathbb{C}[[x]]^{\hat{e}}$. We define the **completion** $\hat{J}$ of J as the image under $\hat{\alpha}$ of the $\mathbb{C}[[x]]^{\hat{e}}$-left ideal generated by $\hat{\alpha}^{-1}(J)$. For simplicity, for elements $f_1, ..., f_m$ in $\mathbb{C}[[x]]$, we shall write $(f_1, ..., f_m)$ for the completion of the two-sided ideal generated by the f_i. A two-sided ideal of the form $(f_1, ..., f_m)$ will be called **finitely generated**.

Proposition 1.6. *The Kernel $\mathcal{K}$ of the abelization* $\text{ab}: \mathbb{C}[[x_1|...|x_n]] \longrightarrow \mathbb{C}[[x_1, \ldots, x_n]]$ *is finitely generated by the commutators $[x_i, x_j] = x_ix_j - x_jx_i$, for $1 \leq i < j \leq n$.*

Proof. We show that for each noncommutative power series f, the difference $f - f_{\text{ab,unab}}$ is in the image under $\hat{\alpha}$ of the $\mathbb{C}[x]^{\hat{e}}$-left ideal generated by the commutators $[x_i, x_j]$, for $i < j$. Without restriction, let f be the sum of its homogeneous components f_k of degree $k \geq 2$. Each difference $f_k - f_{\text{ab,unab},k}$ is of the form $\hat{\alpha}(\sum_{i<j} c_k^{ij} \cdot [x_i, x_j])$, for certain homogeneous $c_k^{i,j}$ in $\mathbb{C}[x]^{\hat{e}}$ of degree $k-2$. Thus $f - f_{\text{ab,unab}}$ is the image under $\hat{\alpha}$ of $\sum_{i<j}(\sum_k c_k^{ij}) \cdot [x_i, x_j]$. $\square$

Locality A family $(h_\alpha)_{\alpha \in A}$ of power series in $\mathbb{C}[[x-p]]$ is called **summable**, if, for each multi-index I, there are only finitely many $\alpha \in A$ such that $h_{\alpha,I} \neq 0$. In this case, we can form the power series

$$\sum_{\alpha \in A} h_\alpha := \sum_I (\sum_\alpha h_{\alpha,I})(x-p)^I.$$

Proposition 1.7. *Both, $\mathbb{C}[[x-p]]$ and $\mathbb{C}\{x-p\}$, are local rings with maximal two-sided ideal $\mathfrak{m}_p$ generated by $x_1 - p_1, \ldots, x_n - p_n$.*

Proof. It suffices to show that each element f of $\mathbb{C}[[x-p]] \setminus \mathfrak{m}_p$ is a left-and right unit. Without restriction, say $f_0 = 1$. Then the family $(1-f)^j; j \geq 0$ is summable. We

have

$$f \cdot \sum_{j=0}^{\infty}(1-f)^j = (1-(1-f))\sum_{j=0}^{\infty}(1-f)^j =$$
$$= \sum_{j=0}^{\infty}(1-f)^j - \sum_{j=1}^{\infty}(1-f)^j = 1.$$

Thus f is a left unit. In the same way, we show that f is a right unit. If f is convergent, the sum $\sum(1-f)^j$ is also convergent. This follows exactly as in the commutative case. □

2 NC complex analytic spaces

NC functions on polydisks

Proposition 2.1. *Fix a point p in $\mathbb{C}^n$. If $f = \sum_I a_I(x-p)^I$ is a noncommutative power series in $\mathbb{C}_r\{x-p\}$ converging on the open polydisk $P(p,r)$ of multi-radius r, then, for each muti-index J with $J \leq I$, the power series*

$$a_J(z) := \sum_{\alpha} a_I(z-p)^{I-_\alpha J},$$

where the sum is taken over all maps $\alpha : J \longrightarrow I$ of multi-indices, converges on $P(p,r)$.

Proof. We must show that, for $q \in P(p,r)$, the series

$$a_J(q) = \sum_I \binom{I}{J} a_I(q-p)^{I-J}$$

converges absolutely. We may assume that all a_I are positive real numbers and that $(p-q)$ is in $\mathbb{R}^n_{\geq 0}$. The product $a_J(q)\cdot(q-p)^J$ lower or equals $\sum_{I\geq J}\binom{\#I}{\#J}a_I(q-p)^I$, which lower or equals $\sum_m^{\infty} m^{\#J}\sum_{\#I=m}a_I(q-p)^I$. The latter converges by the quotient criterium, since we know that f converges absolutely at q. □

The following statement is an immediate consequence of the correponding statement for commutative power series:

Proposition 2.2. *Let $r' \in \mathbb{R}^n_{>0}$ be a mult-radius such that the open polydisk $P(q,r')$ is contained in $P(p,r)$. The power series $f_q := \sum_J a_J(q)(z-q)^J$ converges on $P(q,r')$ and represents the function $f_{\mathrm{ab}}|_{P(q,r')}$.*

Proposition 2.3. *Let p and q be points in $\mathbb{C}^n$ and $r, r' \in \mathbb{R}^n_{>0}$ multi-radius such that $P(q, r')$ is contained in the open polydisk $P(p, r)$. The map*

$$\begin{aligned} \alpha_r(p,q) : \mathbb{C}_r\{x-p\} &\longrightarrow \mathbb{C}_{r'}\{x-q\} \\ \sum_I a_I (x-p)^I &\mapsto \sum_J a_J(q)(x-q)^J, \end{aligned}$$

with $a_J(q)$ as in Proposition 2.1, is an algebra homomorphism.

Proof. Consider two elements $f = \sum_I a_I(x-p)^I$ and $\tilde{f} = \sum_I \tilde{a}_I(x-p)^I$ of $\mathbb{C}_r\{x-p\}$. We have

$$f \cdot \tilde{f} = \sum_K (\sum a_I \cdot \tilde{a}_{\tilde{I}})(x-p)^K,$$

where the sum b_K in the bracket is over all I and $\tilde{I}$ such that $K = (I, \tilde{I})$. We get

$$\alpha_r(p,q)(f \cdot \tilde{f}) = \sum_L b_L(q)(x-q)^L,$$

where $b_L(q)$ is the sum

$$\sum_{K \geq L} \binom{K}{L} b_K (q-p)^{K-L} = \sum_{K \geq L} \sum \binom{K}{L} a_I \cdot \tilde{a}_{\tilde{I}} (q-p)^{K-L}.$$

For a given L, the 4-tuples $(K, \alpha, I, \tilde{I})$, where $\alpha : L \longrightarrow K$ is a map of multi-indices and $I, \tilde{I}$ are multi-indices such that $K = (I, \tilde{I})$, are in one-to-one correspondence with the 6-tuples $(J, \tilde{J}, I, \tilde{I}, \alpha_1, \alpha_2)$, where $J, \tilde{J}$ are multi-indices such that $L = (J, \tilde{J})$ and α_1 : J $\longrightarrow I$ and $\alpha_2 : \tilde{J} \longrightarrow \tilde{I}$ are maps of multi-indices. To the 4-tuple $(K, \alpha, I, \tilde{I})$, assign $(\alpha^{-1}(I), \alpha^{-1}(\tilde{I}), I, \tilde{I}, \alpha|_J, \alpha|_{\tilde{J}})$. To the 6-tuple $(J, \tilde{J}, I, \tilde{I}, \alpha_1, \alpha_2)$, assign $((I, \tilde{I}), (\alpha_1, \alpha_2), I, \tilde{I})$. Observe that, for corresponding tuples, we have $\binom{K}{L} = \binom{I}{J} \cdot \binom{\tilde{I}}{\tilde{J}}$. We can thus write $\alpha_r(p,q)(f \cdot \tilde{f})$ as $\sum_L c_L(x-q)^L$, where

$$c_L = \sum \sum_{I \geq J} \sum_{\tilde{I} \geq \tilde{J}} \binom{I}{J} \cdot \binom{\tilde{I}}{\tilde{J}} a_I \cdot \tilde{a}_{\tilde{I}} (q-p)^{I-J, \tilde{I}-\tilde{J}}.$$

Here, the first sum is over all $J, \tilde{J}$ such that $L = (J, \tilde{J})$. But $\sum_L c_L(x-q)^L = (\sum_J a_J(x-q)^J) \cdot (\sum_{\tilde{J}} \tilde{a}_J (x-q)^{\tilde{J}})$, which is just the product of $\alpha_r(p,q)(f)$ and $\alpha_r(p,q)(\tilde{f})$. $\square$

Example 2.4. Let $n = 2$, $p = 0$ and $f = x_1x_2 - tx_2x_1$. For any $q = (q_1, q_2) \in \mathbb{C}^2$, we get $a_0(q) = q_1q_2 - tq_2q_1 = (1-t)q_1q_2$. $a_{(1)}(q) = a_{(1,2)}q^{(2)} + a_{(2,1)}q^{(2)} = (1-t)q_2$, $a_{(2)}(q) = a_{(1,2)}q^{(1)} + a_{(2,1)}q^{(1)} = (1-t)q_1$, $a_{(1,2)}(q) = a_{(1,2)} = 1$, $a_{(2,1)}(q) = a_{(2,1)} = -t$. Thus, $\sum_J a_J(q)(x-q)^J = (1-t)q_1q_2 + (1-t)q_2(x_1-q_1) + (1-t)q_1(x_2-q_2) + (x_1 - q_1)(x_2-q_2) - t(x_2-q_2)(x_1-q_1)$.

The affine analytic NC space For open subsets $U \subseteq \mathbb{C}^n$, set $\mathcal{O}(U)$ to be the set of all elements $(f_p)_{p\in U}$ of the product $\prod_{p\in U} \mathbb{C}\{x-p\}$ such that the following condition holds: Let p be a point in U and suppose that f_p converges on the open polydisk $P(p,r) \subseteq U$. Then, for each q in $P(p,r)$, we have $f_q = \alpha_r(p,q)(f_p)$. It follows by Proposition 2.3, that $\mathcal{O}(U)$ is an associative ring. In consequence, the assignment

$$\mathcal{O} : U \mapsto \mathcal{O}(U)$$

defines a sheaf of associative $\mathbb{C}$-algebras on $\mathbb{C}^n$. Write $\mathcal{O}_{\mathrm{ab}}$ for the sheaf of holomorphic functions on $\mathbb{C}^n$.

Proposition 2.5. *We have a natural epimorphism* $\mathrm{ab} : \mathcal{O} \longrightarrow \mathcal{O}_{\mathrm{ab}}$ *of sheaves of rings, which splits as a morphism of sheaves of vector spaces.*

Proof. For open subsets U of $\mathbb{C}^n$, we identify $\mathcal{O}_{\mathrm{ab}}(U)$ with set of families $(g_p)_{p\in U}$ of commuative power series $g_p \in \mathbb{C}\{x_1 - p_1, \dots, x_n - p_n\}$, such that the functions represented by g_p and g_q coincide on the intersection of the convergence regions of g_p and g_q (if g_p converges absolutely at q this is equivalent to $\alpha_r(p,q)(g_{p,\mathrm{unab}}) = g_{q,\mathrm{unab}}$). Now, $\mathrm{ab}(U)$ maps an element $(f_p)_{p\in U}$ of $\mathcal{O}(U)$ to the family $(f_{p,\mathrm{ab}})$. We even get that $\mathcal{O}(U)$ is surjective, for any open subset of $\mathbb{C}^n$, since $\mathcal{O}(U)$ splits by the map $g = (g_p)_{p\in U} \mapsto (g_{p,\mathrm{unab}})_{p\in U}$. □

By the following proposition, together with Proposition 1.7, the pair $(\mathbb{C}^n, \mathcal{O})$ is a locally associative ringed space.

Proposition 2.6. *For each point $p \in \mathbb{C}^n$, the stalk $\mathcal{O}_p$ is isomorphic to the noncommutative convergent power series ring $\mathbb{C}\{x-p\}$.*

Proof. Exercise. □

Local models Now we are able to define local models of NC analytic spaces. Let U be an open subset of $\mathbb{C}^n$. Let $\mathcal{O}_U$ be the restriction of $\mathcal{O}$ to U. Let $\mathcal{I}$ be a sheaf of twosided ideals of $\mathcal{O}_U$, such that $\mathcal{I}_{\mathrm{ab}}$ is a coherent $\mathcal{O}_{U,\mathrm{ab}}$-module. Let V be the subset of U given by the zeros of $\mathcal{I}_{\mathrm{ab}}$. Write ι for the inclusion $V \longrightarrow U$. Set $\mathcal{O}_V := \iota^{-1}(\mathcal{O}_U/\mathcal{I})$.

Proposition 2.7. *For $q \in V$, the stalk $\mathcal{O}_{V,q}$ is naturally isomorphic to the quotient of $\mathbb{C}\{x-q\}$ by the twosided ideal by $\mathcal{I}_q$.*

In consequence, the pair $V(\mathcal{I}) := (V, \mathcal{O}_V)$ is a locally associative ringed space. Pairs of this form will be called **local models of NC complex analytic spaces**. Observe that the abelization $(V, \mathcal{O}_{V,\mathrm{ab}})$ is a local model of a complex analytic space in the classical sense.

Example 2.8. Let $\mathcal{K}$ be the kernel of the sheaf homomorphism $\mathrm{ab} : \mathcal{O}_U \longrightarrow \mathcal{O}_{U,\mathrm{ab}}$. Then $\mathcal{K}_{\mathrm{ab}} = 0$ is coherent and the pair $V(\mathcal{I})$ is just a local model for a (commutative) analytic manifold.

NC complex spaces Now, we come to the main definition.

Definition 2.9. An **NC complex analytic space** is a locally associative ringed space $(X, \mathcal{O}_X)$, locally isomorphic to a local model. An **NC complex analytic manifold** is a locally associative ringed space, locally isomorphic to an open subspace of $\mathbb{C}^n$ with the canonical NC structure.

3 Examples

Example 3.1. (Local models)
Choose any splitting $s : \mathcal{O}_{\mathrm{ab}} \longrightarrow \mathcal{O}$ of the abelization map in the category of sheaves of $\mathbb{C}$-vector spaces. For any local model $X_{\mathrm{ab}} = V(U, \mathcal{I}_{\mathrm{ab}})$ of a complex analytic space, set $\mathcal{I}$ to be the two-sided ideal $(s\mathcal{I})^\wedge$, generated by the image of $\mathcal{I}_{\mathrm{ab}}$ under s. (To generate more examples, add subsheaves of the commutator sheaf to $\mathcal{I}$.) We have $\mathrm{ab}(\mathcal{I}) = \mathcal{I}_{\mathrm{ab}}$. Then the NC local model $X = V(U, \mathcal{I})$ has the same underlying topological space as X_{ab} and $\mathcal{O}_{X_{\mathrm{ab}}} = \mathrm{ab}(\mathcal{O}_X)$.

Example 3.2. (Analytic supermanifolds)
By definition, an analytic supermanifold is a locally ringed space $(M, \mathcal{O}_M)$ with a $\mathbb{Z}/2$-graded structure sheaf $\mathcal{O}_M = \mathcal{O}_M^0 \oplus \mathcal{O}_M^1$ such that the following two conditions hold:

(1) Let $\mathcal{O}_M^{\mathrm{red}} := \mathcal{O}_M/(\mathcal{O}_M^1)$ be the quotient of $\mathcal{O}_M$ modulo the two-sided ideal, generated by $\mathcal{O}_M^1$. The pair $(M, \mathcal{O}_M^{\mathrm{red}})$ is a usual analytic manifold.

(2) There is an $r \geq 0$ and each point of M has an open neighbourhood P in M such that the restriction $\mathcal{O}_M|_P$ of the structure sheaf is isomorphic to the exterior $\mathcal{O}_M^{\mathrm{red}}|_P$-algebra over the free $\mathcal{O}_M^{\mathrm{red}}|_P$-module of rank r.

Each analytic supermanifold is an NC complex analytic space: Let $M^{\mathrm{red}} = (M, \mathcal{O}_M^{\mathrm{red}})$ be n-dimensional. Choose an analytic atlas of M^{red} of the form

$$\{\phi_k : P(p^{(k)}, r^{(k)}) \longrightarrow P^{(k)} |\; k \in K\},$$

where $(P^{(k)})_{k \in K}$ is an open covering of M such that the restriction $\mathcal{O}_M|_{P^{(k)}}$ satisfies condition (2). Consider the sheaf $\mathcal{I}$ of two-sided ideals of the NC structure sheaf $\mathcal{O}_{P(p^{(k)}, r^{(k)}) \times \mathbb{C}^r}$ of $P(p^{(k)}, r^{(k)}) \times \mathbb{C}^r$, generated by the following relations:

$$\begin{aligned} [x_i - p_i^{(k)}, x_j - p_j^{(k)}] \quad & \text{for} \quad 1 \leq i < j \leq n, \\ [x_i - p_i^{(k)}, y_j] \quad & \text{for} \quad i = 1, \ldots, n \text{ and } j = 1, \ldots, r, \\ y_i y_j + y_j y_i \quad & \text{for} \quad 1 \leq i \leq j \leq r. \end{aligned}$$

Observe that the abelization $\mathcal{I}_{\mathrm{ab}}$ is coherent, and we have an isomorphism

$$(P^{(k)}, \mathcal{O}_M|_{P^{(k)}}) \cong V(P(p^{(k)}, r^{(k)}), \mathcal{I}).$$

Example 3.3. (NC projective space)
The (commutative) projective space $\mathbb{C}P^2$ is obtained by glueing three copies U_1, U_1 and U_2 of $\mathbb{C}^2$ along the following isomorphisms between open subsets:

$$\begin{aligned} U_{01} &\longrightarrow U_{10}, (x_1, x_2) \mapsto (\tfrac{1}{x_1}, \tfrac{x_2}{x_1}), \\ U_{02} &\longrightarrow U_{20}, (x_1, x_2) \mapsto (\tfrac{1}{x_2}, \tfrac{x_1}{x_2}), \\ U_{12} &\longrightarrow U_{21}, (y_1, y_2) \mapsto (\tfrac{y_1}{y_2}, \tfrac{1}{y_2}), \end{aligned}$$

where U_{ij} is the subset of U_i, where the indicated map is defined. For each point $p_1 \neq 0$ in $\mathbb{C}$, the fraction $\frac{1}{x_1}$ can be interpreted as a power series in $x_1 - p_1$, converging near the point p_1. In order to use the same identifications to glue the noncommutative affine spaces U_0, U_1 and U_2, we have to choose, if, for (p_1, p_2) in $\mathbb{C}^2$ with $P_1 \neq 0$, we interpret $\frac{x_2}{x_1}$ as the noncommutative power series $x_2 \cdot \frac{1}{x_1}$ in $x_1 - p_1, x_2 - p_2$ or as the noncommutative power series $\frac{1}{x_1} \cdot x_2$. If, by convention, we always use the second interpretation, we get compatibility of the gluing maps. To be precise, the isomorphism $\phi_{10} : (U_{01}, Oh_{U_{01}} \longrightarrow (U_{10}, \mathcal{O}_{U_{10}}$ of NC complex spaces is given as follows: The map of topolological spaces $U_{01} \longrightarrow U_{10}$ is as in the commutative case. The associated map $\mathcal{O}_{U_{10}} \longrightarrow \phi_{10,*}\mathcal{O}_{U_{01}}$ is stalkwise given by the map $\mathbb{C}\{y - q\} \longrightarrow \mathbb{C}\{x - p\}$, sending $y_1 - q_1$ to the power series $\sum_I a_{1,I}(x - p)^I$, where

$$a_{1,I} = \begin{cases} \frac{-1}{p_1^{n+1}} & for \quad I = (1, \ldots, 1) \quad (n\text{copies}) \\ 0 & else \end{cases}$$

and sending $y_2 - q_2$ to the power series $\sum_I a_{2,I}(x - p)^I$, where

$$a_{2,I} = \begin{cases} \frac{-1}{p_1^{n+1}} \cdot p_2 & for \quad I = (1, \ldots, 1) \quad (n \text{ copies}) \\ \frac{-1}{p_1^{n+1}} & for \quad I = (1, \ldots, 1, 2) \quad (n \text{ copies of } 1) \\ 0 & else \end{cases}$$

We leave it to the reader to give an explicit description of the other glueing isomorphisms. In this way, we define an NC structure on the topological space $\mathbb{C}P^2$. Exactly in the same way, we get an NC structure $\mathcal{O}_{\mathbb{C}P^n}$ on $\mathbb{C}P^n$, for each $n \geq 2$. The NC complex manifold $(\mathbb{C}P^n, \mathcal{O}_{\mathbb{C}P^n})$ is called **NC projective space**. We shall denote it by $\mathbb{C}P^n_{NC}$. We recover the ordinary, commutative projective space as abelization of the NC complex space.

Example 3.4. (NC projective varieties)
By Chow's theorem, each complex analytic subspace X of $\mathbb{C}P^n$ is algebraic, i.e. given by homogeneous polynomials $f_1, \ldots, f_m$ in $\mathbb{C}[x_0, \ldots, x_n]$. For each $i = 1, \ldots, m$, we can choose a homogeneous lift of f_i in $\tilde{f}_i \in \mathbb{C}[x_0|...|x_n]$ and consider the two-sided ideal I of $\mathbb{C}[x_0|...|x_n]$ generated by those lifts.
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